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On the Weyl-Wigner-Moyal Description of SU(∞) Nahm Equations

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ABSTRACT

We show how the reduced Self-dual Yang-Mills theory described by the Nahm equations can be carried over to the Weyl-Wigner-Moyal formalism employed recently in Self-dual gravity. Evidence of the existence of correspondence between BPS magnetic monopoles and space-time hyper-Kähler metrics is given.

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1. **Introduction**

BPS magnetic monopoles have attracted a great deal of interest in the last two years. The surprising realization of electromagnetic duality in some effective $n = 2$ supersymmetric Yang-Mills theories in four dimensions opens various windows for the understanding the non-perturbative aspects of quantum field theories of point particles and strings [1]. BPS monopoles are solutions of Bogomolny equations and are particular solutions to the static, finite energy Yang-Mills-Higgs equations. Many authors working on the "mathematical side" have shown the correspondence of BPS monopoles with algebraic geometry and twistor theory [2]. Moreover, Nahm has codified these monopole solutions in terms of a modified version of Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction [3]. There is a one to one correspondence between solutions of Bogomolny equations with appropriate asymptotic boundary conditions and solutions of SU(2) Nahm equations, $\frac{dT_i}{ds} = \frac{1}{2}\epsilon_{ijk}[T_j, T_k]$, where summation over repeated indices is assumed [2]; (the matrices T_i must, of course, satisfy some consistence conditions). These mathematical constructions have been recently interpreted in terms of Dirichlet (D)-3-branes realization of $D = 4$ $n = 2$ super Yang-Mills theory in type IIB superstrings [4]. Application of D-instantons to ADHM construction in terms of monads was presented in Ref. [5]. Furthermore the $\mathrm{SL}(2, \mathbf{Z})$ duality symmetry of type IIB superstrings was shown to be very useful tool to study some moduli spaces of BPS monopoles and to explain, among another subjects, the origin of the"mirror symmetry" in three dimensions [6].

On the other hand, Nahm equations also appears in the Ashtekar's approach to $3 + 1$ formulation of complex Einstein's equations. Moreover the self-dual metrics are in correspondence with solutions to Nahm equations for a triad of Hamiltonian vector fields [7]. It is very well known that Moduli space of BPS-saturated configurations with magnetic charge k, $\mathcal{M}_k$, has hyper-Kähler structure [2]. Nevertheless not apparent correspondence exists with the space-time hyper-Kähler geometry. The natural question arises: Is it possible to relate a BPS monopole with a space-

time hyper-Kähler four dimensional metric? In this paper we show that the that answer to this question, in some cases, should be yes. In order to do that we will employed the Weyl-Wigner-Moyal (WWM) formalism of quantum mechanics. This formalism has been recently applied to the self-dual Yang-Mills and the principal chiral model approaches to self-dual gravity [8,9,10]. In particular, in the last reference of [10], some explicit solutions were obtained. Then it has been found in [11], how some general properties of two-dimensional classical chiral models can be carried over to self-dual gravity. Furthermore, Uhlenbeck's unitons, [12] which are solutions to chiral equations (with appropriate boundary conditions), are found to be connected with hyper-Kähler metrics, defining the "gravitational uniton".

The study of some large-N limits of various physical systems shows that radical simplifications occur in their mathematical structures. Some two-dimensional integrable systems seem to be greatly affected in this limit, turning out 'more integrable'. Example of this 'induced integrability' occurs in the large-N limit of SU(N) Nahm equations [13]. Following Witten [14], one would imitate Ward's procedure of SU(∞) Nahm equation results, to get integrability, for instance, in full Yang-Mills theory. Here is where WWM-formalism might be of some interest in order to address this problem.

Besides of self-dual gravity we find in large-N limit of BPS monopole equations another example where the WWM-formalism can be applied.

The paper is organized as follows. In Section 2 we review the large-N limit of Nahm equations given in Refs. [13,15]. Section 3 is devoted to find the Moyal deformation of Nahm equations using the WWM-formalism. In Section 4 we find some solutions of the Moyal deformation of Nahm equations and for the SU(∞) case as well. Also we comment about the possible relation between BPS monopoles and particular hyper-Kähler metrics. Finally in Section 5 we give our final remarks.

2. **Preliminary of Nahm Equations**

2.1. SU(N) Nahm Equations

In this section we briefly describe the geodesics on SU(N). Basically we follows Ward's papers [13] and the finite dimensional group version of the description of self-dual membranes in (4+)1-dimensions by using self-dual Yang-mills theory on the four-dimensional Euclidean space-time [15].

Let $\mathbf{G}$ be an d-dimensional Lie group and $\mathcal{G}$ its corresponding Lie algebra. Let $A^j(t) \in \mathrm{so}(3) \otimes \mathcal{G}$ be a vector field on $\mathbf{G}$ and j is a so(3) index. In the temporal gauge connection, $A_0 = 0$, full Yang-Mills equations are reduced to a system with Lagrangian [15]

$$L_N = \alpha \mathrm{Tr}\left\{ A^j_{,t} A^j_{,t} + \frac{1}{2}[A^j, A^k][A^j, A^k] \right\}, \tag{1}$$

where $A^j_{,t} \equiv \frac{\partial A^j}{\partial t}$, etc., α is a constant and 'Tr' stands for a bilinear invariant form on the Lie algebra $\mathcal{G}$ which we will assume to be semi-simple, and the constraint

$$[A^j_{,t}, A^j] = 0, \tag{2}$$

which is the Gauss law.

The equations of motion of (1) are

$$A^j_{,tt} + [A^k, [A^k, A^j]] = 0. \tag{3}$$

¿From the Lagrangian (1) the 'conserved energy' of this system is

$$E_N = \mathrm{Tr}\left\{ A^j_{,t} A^j_{,t} - \frac{1}{2}[A^j, A^k][A^j, A^k] \right\}, \tag{4}$$

which is not positive-definite.

On the other hand it is easy to see that Eqs. (2) and (3) are implied by the reduced version of self-dual Yang-Mills equations

$$A^j_{,t} = \frac{1}{2}\epsilon^{jkl}[A^k, A^l]. \tag{5}$$

These equations are the famous Nahm equations and their solutions imply vanishing of energy in (4).

2.2. SU(∞) Nahm Equations

The transition from SU(N) to the SU(∞) gauge theory involves the substitution, in the place of the Lie algebra su(N), the area-preserving diffeomorphisms algebra (or Poisson bracket algebra) sdiff(Σ), on a two-dimensional manifold (Σ, ω) [16,17,18]. Here Σ is a two-dimensional simply connected symplectic manifold with local real coordinates $\{p, q\}$. Locally, by the Darboux's theorem, the symplectic structure is given by $\omega = dp \wedge dq$. sdiff(Σ) is precisely the Lie algebra associated with the infinite dimensional Lie group, SDiff(Σ), which is the group of diffeomorphisms on Σ preserving the symplectic structure.

The Hamiltonian vector fields are $\mathcal{U}_{\mathcal{H}_i} = \omega^{-1}(d\mathcal{H}_i)$ satisfying the sdiff(Σ) algebra $[\mathcal{U}_{\mathcal{H}_i}, \mathcal{U}_{\mathcal{H}_j}] = \mathcal{U}_{\{\mathcal{H}_i,\mathcal{H}_j\}_P}$, for all $(i \neq j)$. Here $\{,\}_P$ stands for the Poisson bracket. Locally it can be written as $\{\mathcal{H}_i, \mathcal{H}_j\}_P = \omega^{-1}(d\mathcal{H}_i, d\mathcal{H}_j) = \omega^{lm}\partial_l\mathcal{H}_i\partial_m\mathcal{H}_j$, where $\partial_l \equiv \frac{\partial}{\partial x^l}$, $(l = 0, 1)$, $x^0 = p$, $x^1 = q$ and $\mathcal{H}_i = \mathcal{H}_i(p, q)$. The generators of sdiff(Σ) are the Hamiltonian vector fields $\mathcal{U}_{\mathcal{H}_i} = \frac{\partial\mathcal{H}_i}{\partial q}\frac{\partial}{\partial p} - \frac{\partial\mathcal{H}_i}{\partial p}\frac{\partial}{\partial q}$, associated to the Hamiltonian functions $\mathcal{H}_i$.

Considering now the case when $N \to \infty$. Then the corresponding Lie algebra $\mathcal{G} = \mathrm{su}(\infty)$ should be replaced by the Poisson bracket algebra on Σ [19]

$$\mathrm{su}(\infty) \cong \mathrm{sdiff}(\Sigma). \tag{6}$$

Then one can take A^j to be a triad of Hamiltonian fields

$$A^j = \frac{\partial \mathcal{A}^j}{\partial q}\frac{\partial}{\partial p} - \frac{\partial \mathcal{A}^j}{\partial p}\frac{\partial}{\partial q}, \tag{7}$$

where $\mathcal{A}^j = \mathcal{A}^j(t, p, q) \in \mathrm{so}(3) \otimes \mathrm{sdiff}(\Sigma)$.

Now following the standard prescriptions in the large-N limit in gauge theories

$$\frac{(2\pi)^4}{N^3}\mathrm{Tr}(...) \Rightarrow -\int_\Sigma (...) dpdq,$$

$$A^j \Rightarrow \mathcal{A}^j, \tag{8}$$

$$[A^j, A^k] \Rightarrow \{\mathcal{A}^j, \mathcal{A}^k\}_P$$

and taking $\alpha = \frac{(2\pi)^4}{N^3}$, the large-$N$ limit equation of motion (3) is

$$\mathcal{A}^j_{,tt} + \{\mathcal{A}^k, \{\mathcal{A}^k, \mathcal{A}^j\}_P\}_P = 0, \tag{9}$$

and the constraint (2) reads

$$\{\mathcal{A}^j_{,t}, \mathcal{A}^j\}_P = 0. \tag{10}$$

Eqs. (9) and (10) can be derived from the Lagrangian

$$L_N^\infty = -\int_\Sigma dpdq \Big\{ \mathcal{A}^j_{,t}\mathcal{A}^j_{,t} + \frac{1}{2}\{\mathcal{A}^j, \mathcal{A}^k\}_P\{\mathcal{A}^j, \mathcal{A}^k\}_P \Big\}. \tag{11}$$

The large-N limit of 'conserved energy' is

$$E_N^\infty = -\int_\Sigma dpdq \Big\{ \mathcal{A}^j_{,t}\mathcal{A}^j_{,t} - \frac{1}{2}\{\mathcal{A}^j, \mathcal{A}^k\}_P\{\mathcal{A}^j, \mathcal{A}^k\}_P \Big\}. \tag{12}$$

Thus the large-N limit of Nahm equations is

$$\mathcal{A}^j_{,t} = \frac{1}{2}\epsilon^{jkl}\{\mathcal{A}^k, \mathcal{A}^l\}_P, \tag{13}$$

which are precisely the large-N limit of the self-duality condition. One can see that once again SU(∞) Nahm equations (13) imply equations (9) and (10).

3. The Moyal Deformation of Nahm Equations

The aim of this section is to show that the reduced self-dual Yang-Mills theory described by Nahm equations, as we saw in the above section, can be carried over into the Weyl-Wigner-Moyal (WWM)-formalism. We will show that the large-N limit can be also achieved by taking the $\hbar \to 0$ limit.

The operator-valued equations (operator analog of Eq. (3)) for $\hat{A}^j \in \mathrm{so}(3) \otimes \hat{\mathcal{U}}$ where $\hat{\mathcal{U}}$ is the Lie algebra of anti-self-dual operators acting on the Hilbert space $H = L^2(\mathbf{R})$, are

$$\hat{A}^j_{,tt} + [\hat{A}^k, [\hat{A}^k, \hat{A}^j]] = 0. \tag{14}$$

While the quantum Gauss law is written as

$$[\hat{A}^j_{,t}, \hat{A}^j] = 0, \tag{15}$$

where $[,]$ is the commutator.

First of all define

$$\hat{\mathcal{A}} := i\hbar A^j. \tag{16}$$

With Eq. (16), the Eq. (14) can be rewritten as

$$\hat{\mathcal{A}}^j_{,tt} + \frac{1}{i\hbar}[\hat{\mathcal{A}}^k, \frac{1}{i\hbar}[\hat{\mathcal{A}}^k, \hat{\mathcal{A}}^j]] = 0, \tag{17}$$

and the constraint

$$\frac{1}{i\hbar}[\hat{\mathcal{A}}^j_{,t}, \hat{\mathcal{A}}^j] = 0. \tag{18}$$

By simple calculations we find that equation of motion (17) can be derived from the 'quantum Lagrangian'

$$L^{(q)} := Tr\Big\{2\pi\hbar\Big[\hat{\mathcal{A}}^j_{,t}\hat{\mathcal{A}}^j_{,t} - \frac{1}{2\hbar^2}[\hat{\mathcal{A}}^j, \hat{\mathcal{A}}^k][\hat{\mathcal{A}}^j, \hat{\mathcal{A}}^k]\Big]\Big\}$$

$$= 2\pi\hbar\sum_j < \psi_j|\hat{\mathcal{A}}^j_{,t}\hat{\mathcal{A}}^j_{,t} - \frac{1}{2\hbar^2}[\hat{\mathcal{A}}^j, \hat{\mathcal{A}}^k][\hat{\mathcal{A}}^j, \hat{\mathcal{A}}^k]\Big]|\psi_j >, \tag{19}$$

where 'Tr' is the sum over diagonal elements with respect to an orthonormal basis $\{|\psi_j >\}_{j\in\mathbf{N}}$ in $L^2(\mathbf{R})$ *i.e.*

$$< \psi_j|\psi_k >= \delta_{jk}, \qquad \sum_j |\psi_j >< \psi_j| = \hat{I}. \tag{20}$$

Operator equation of motion (17) and constraint (18) are implied by the operator-valued Nahm equations

$$\hat{\mathcal{A}}^j_t(t) = \frac{1}{2i\hbar}\epsilon^{jkl}[\hat{\mathcal{A}}^k(t), \hat{\mathcal{A}}^l(t)]. \tag{21}$$

The *Weyl correspondence* $\mathcal{W}^{-1}$ establishes a one to one correspondence between self-adjoint linear operators $\mathcal{B}$ acting on Hilbert space $H = L^2(\mathbf{R})$ and the space of real smooth functions $C^\infty(\Sigma, \mathbf{R})$ on the phase space manifold Σ. This correspondence $\mathcal{W}^{-1} : \mathcal{B} \to C^\infty(\Sigma, \mathbf{R})$, is given by

$$\mathcal{A}^j(t,p,q;\hbar) \equiv \mathcal{W}^{-1}(\hat{\mathcal{A}}^j) := \int\limits_{\infty}^{\infty} < q - \frac{\xi}{2}|\hat{\mathcal{A}}^j(t)|q + \frac{\xi}{2} > \exp\big(\frac{i}{\hbar}\xi p\big) d\xi, \tag{22}$$

for all $\hat{\mathcal{A}} \in \mathcal{B}$ and $\mathcal{A} \in C^{\infty}(\Sigma, \mathbf{R})$. Define the Moyal product '$\star$' on $C^{\infty}(\Sigma, \mathbf{R})$ to be $\mathcal{H}_i \star \mathcal{H}_j := \mathcal{H}_i \exp(\frac{i\hbar}{2} \overleftrightarrow{\mathcal{P}})\mathcal{H}_j$, $(i \neq j)$ where $\overleftrightarrow{\mathcal{P}} := \frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}$, $\mathcal{H}_i = \mathcal{H}_i(p,q;\hbar)$.

One can see easily that $\mathcal{W}^{-1}$ is a Lie algebra isomorphism $\mathcal{W}^{-1} : \big(\mathcal{B}, [,]\big) \rightarrow \big(\mathcal{M}, \{\cdot,\cdot\}_M\big)$, and it is given by

$$\mathcal{W}^{-1}\big(\frac{1}{i\hbar}[\hat{\mathcal{H}}_i, \hat{\mathcal{H}}_j]\big) = \frac{1}{i\hbar}(\mathcal{H}_i \star \mathcal{H}_j - \mathcal{H}_j \star \mathcal{H}_i) \equiv \{\mathcal{H}_i, \mathcal{H}_j\}_M, \tag{23}$$

where $\{,\}_M$ is the Moyal bracket (for details and conventions see Ref. [10]) and $\mathcal{M}$ is called *the Moyal algebra.*

Moyal algebra $\mathcal{M}$ is isomorphic to a deformation of Poisson algebra sdiff(Σ) *i.e.* $\mathcal{M} \cong \text{diff}_{\hbar}(\Sigma)$, where $\hbar$ is the deformation parameter. The correspondence with the Poisson algebra is given by taking the limit $\hbar \rightarrow 0$ *i.e.* $\lim_{\hbar\rightarrow 0} \text{sdiff}_{\hbar}(\Sigma) = \text{sdiff}(\Sigma)$, or

$$\lim_{\hbar\rightarrow 0} \mathcal{H}_i \star \mathcal{H}_j = \mathcal{H}_i\mathcal{H}_j \quad \text{and} \quad \lim_{\hbar\rightarrow 0}\{\mathcal{H}_i, \mathcal{H}_j\}_M = \{\mathcal{H}_i, \mathcal{H}_j\}_P. \tag{24}$$

¿From the WWM-formalism an easy computation shows that operator equation of motion (17) yields

$$\mathcal{A}^j_{,tt} + \{\mathcal{A}^j, \{\mathcal{A}^j, \mathcal{A}^k\}_M\}_M = 0, \tag{25}$$

and the associated constraint is

$$\{\mathcal{A}^j_{,t}, \mathcal{A}^j\}_M = 0. \tag{26}$$

Eq. (25) is the Moyal deformation of Eq. (3). A computation shows that Eq. (25) can be derived from the Lagrangian

$$L_N^{(M)} := -\int_\Sigma dpdq \Big\{ \mathcal{A}^j_{,t} \star \mathcal{A}^j_{,t} + \frac{1}{2}\{\mathcal{A}^j, \mathcal{A}^k\}_M \star \{\mathcal{A}^j, \mathcal{A}^K\}_M \Big\}. \tag{27}$$

While the Moyal deformed 'conserved energy' is

$$E_N^{(M)} = -\int_\Sigma dpdq \Big\{ \mathcal{A}^j_{,t} \star \mathcal{A}^j_{,t} - \frac{1}{2}\{\mathcal{A}^j, \mathcal{A}^k\}_M \star \{\mathcal{A}^j, \mathcal{A}^k\}_M \Big\}. \tag{28}$$

Using the properties (24) we find the result

$$\lim_{\hbar\to 0} L_N^{(M)} = L_N^\infty, \quad \lim_{\hbar\to 0} E_N^{(M)} = E_N^\infty. \tag{29}$$

Of course we have assumed that the fields $\mathcal{A}^j(t,p,q,\hbar)$ can be written as [8]

$$\mathcal{A}^j(t,p,q;\hbar) = \sum_{k=0}^{\infty} \hbar^k \mathcal{A}^j_k(t,p,q). \tag{30}$$

and $\lim_{\hbar\to 0} \mathcal{A}^j(t,p,q;\hbar) = \mathcal{A}^j_0(t,p,q)$.

Using Weyl correspondence described as above we get a triad of functions $\mathcal{A}^j = \mathcal{A}^j(t,p,q;\hbar) \in \mathrm{so}(3) \otimes \mathcal{M}$ which satisfy the Moyal deformation of Nahm equations

$$\mathcal{A}^j_{,t} = \frac{1}{2}\epsilon^{jkl}\{\mathcal{A}^k, \mathcal{A}^l\}_M. \tag{31}$$

Using correspondence (24) we finally obtain the well known SU(∞) Nahm equations [13]

$$\mathcal{A}^j_{0,t} = \frac{1}{2}\epsilon^{jkl}\{\mathcal{A}^k_0, \mathcal{A}^l_0\}_P, \tag{32}$$

or after some minor manipulations

$$\mathcal{A}^j_{0,t} = \epsilon^{jkl}\mathcal{A}^k_{,q}\mathcal{A}^l_{,p}. \tag{33}$$

In the next section we shall intend to obtain some solutions of the above equation (33) using WWM-formalism.

Lax Pair

Finally we find the Lax pair for the Moyal deformation of Nahm equations (31). In order to do this, in analogy of [13], we consider the following system of linear partial differential equations on the function $E = E(\lambda, t, p, q) \in C^\infty(\mathbf{C}^* \times T \times \Sigma)$, where T is isomorphic to the real line $\mathbf{R}$, $\mathbf{C}^* = \mathbf{C} - \{0\}$, and $\Sigma \subset \mathbf{R}^2$

$$i\hbar\partial_t E_\lambda = (-a^3 - \lambda a^-) \star E_\lambda, \tag{34a}$$

$$i\hbar\partial_t E_\lambda = (a^3 + \lambda^{-1} a^+) \star E_\lambda, \tag{34b}$$

where $\lambda \in \mathbf{C}^*$, $E_\lambda = E_\lambda(t,p,q) := E(\lambda,t,p,q)$ and $a^\pm := a^1 \pm ia^2$ with $a^i = a^i(t,p,q)$, $(i = 1,2,3)$ independent of λ.

The integrability conditions of the above linear system read

$$\lambda^0 \mapsto \ a^3_{,t} = \frac{1}{2}\{a^+, a^-\}_M, \tag{35a}$$

$$\lambda^1 \mapsto \ a^-_{,t} = \{a^3, a^-\}_M, \tag{35b}$$

$$\lambda^{-1} \mapsto \ a^+_{,t} = \{a^+, a^3\}_M. \tag{35c}$$

Thus one can find that Eqs. (35) written in terms of the a^i's, $i = 1, 2, 3$, are exactly the Moyal deformation of Nahm equations, (31). Therefore *the system (34) constitutes the Lax pair of (31).*

4. **Searching For Solutions**

In this section we attempt searching for some solutions to the Moyal deformation and SU(∞) Nahm equations (13). In order to do that we will follow the same method which we have used to construct some solutions to Park-Husain heavenly equations in the last reference of [10]. The method was explained in [10,11] and it not will be repeated here.

Let $\Phi : \mathcal{G}_{\mathbf{C}} \rightarrow \hat{\mathcal{I}}$ be a Lie algebra homomorphism. Here $\mathcal{G}_{\mathbf{C}} = \mathcal{G} \otimes \mathbf{C}$ and $\hat{\mathcal{I}}$ is the complex Lie algebra of operators. Now we will apply the explicit Lie algebra homomorphism Φ for the cases of $\mathcal{G}_{\mathbf{C}} = \mathrm{su}(2)$ and $\mathcal{G} = \mathrm{sl}(2, \mathbf{R})$ described in Ref. [20].

Elliptic Curves

The solutions of sl(2)-Nahm's equations (5) were obtained some years ago [21,22,23]. Basically these solutions are solutions of the Euler equations [23]. Some of these solutions are expressed in terms of the theory of elliptic curves. These are

$$A^j(t) = -\sqrt{\wp(t) - e_j}, \tag{36}$$

where $\wp(t)$ is the Weierstrass function on the elliptic curve (with semi-periods Ω_1, Ω_2 and $\Omega_3 = \Omega_1 + \Omega_2$) determined by the two first integrals $I_1 = (A^1)^2 - (A^2)^2 = e_2 - e_1$ and $I_2 = (A^1)^2 - (A^3)^2 = e_3 - e_1$. The fact that $A^j \in \mathrm{so}(3) \otimes \mathrm{sl}(2)$ implies that A^j_a is of the from

$$A^j = A^j(t) = A^j_a(t)\tau_a, \quad a = 1, 2, 3 \tag{37}$$

and should be a diagonal matrix

$$\begin{pmatrix} -\sqrt{\wp(t) - e_1} & 0 & 0 \\ 0 & -\sqrt{\wp(t) - e_2} & 0 \\ 0 & 0 & -\sqrt{\wp(t) - e_3} \end{pmatrix}. \tag{38}$$

Thus solutions of Nahm equations are given by

$$A^j(t) = \sqrt{\wp(t) - e_j}\ \tau_j. \tag{39}$$

Applying the Lie algebra homomorphism $\Phi : \mathrm{su}(2) \to \hat{\mathcal{I}}$ we get

$$\hat{\mathcal{A}}^j = \hat{\mathcal{A}}^j(t) = -i\hbar\sqrt{\wp(t) - e_j}\ \hat{\mathcal{X}}_j \in \mathrm{so}(3) \otimes \hat{\mathcal{I}} \tag{40}$$

where $\hat{\mathcal{X}}_j \equiv \Phi(\tau_j)$.

Using WWM-formalism [10] we find that if Eq. (40) is solution of (17), then

$$\mathcal{A}^j(t,p,q;\hbar) = -\sqrt{\wp(t) - e_j}\ \mathcal{X}_j(p,q;\hbar), \tag{41}$$

where

$$\mathcal{X}_j(p,q;\hbar) := \int\limits_{\infty}^{\infty} < q - \frac{\xi}{2}|\hat{\mathcal{X}}_j|q + \frac{\xi}{2} > \exp\left(\frac{i}{\hbar}\xi p\right) d\xi, \tag{42}$$

should be solution of the Moyal deformation of Nahm equation (31).

<u>The SU(2) Magnetic Monopole</u>

SU(2) BPS monopoles can be seen to be solutions of su(2) Nahm equations and can be written as

$$A^j = A^j(t) = A^j_a(t)\tau_a, \quad a = 1,2,3 \tag{43}$$

where

$$\tau_1 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{44}$$

satisfy as usual

$$[\tau_a, \tau_b] = \epsilon_{abc}\tau_c.$$

The matrix $A^j_a(t)$ is a diagonal matrix [21,22,23]

$$\begin{pmatrix} \mathrm{ns}(t,k) & 0 & 0 \\ 0 & \mathrm{ds}(t,k) & 0 \\ 0 & 0 & \mathrm{cs}(t,k) \end{pmatrix}. \tag{45}$$

where 'ns', 'ds' and 'cs' are elliptic functions.

Using the results of Refs. [10,20]

$$\Phi(\tau_1) = \hat{\mathcal{X}}_1 := i\beta\hat{q} + \frac{1}{2\hbar}(\hat{q}^2 - 1)\hat{p}, \tag{46a}$$

$$\Phi(\tau_2) = \hat{\mathcal{X}}_2 := -\beta\hat{q} + \frac{i}{2\hbar}(\hat{q}^2 + 1)\hat{p}, \tag{46b}$$

$$\Phi(\tau_3) = \hat{\mathcal{X}}_3 := -i\beta\hat{1} - \frac{1}{\hbar}\hat{q}\hat{p}, \tag{46c}$$

where $\beta \in \mathbf{R}$ is any constant; $\hat{q}$ and $\hat{p}$ are the position and momentum operators, respectively. Then, the complex functions $\mathcal{A}^j = \mathcal{A}^j(t,p,q)$ should be a solution of the Moyal deformation of the Nahm equations.

$$\mathcal{A}^1(t,p,q;\hbar) = \mathrm{ns}(t,k)[\frac{i}{2}p(q^2-1) - \hbar q(\beta + \frac{1}{2})], \tag{47a}$$

$$\mathcal{A}^2(t,p,q;\hbar) = \mathrm{ds}(t,k)[-\frac{1}{2}p(q^2+1) - i\hbar q(\beta + \frac{1}{2})], \tag{47b}$$

$$\mathcal{A}^3(t,p,q;\hbar) = \mathrm{cs}(t,k)[-ipq + \hbar(\beta + \frac{1}{2})]. \tag{47c}$$

Taking the $\hbar \to 0$ limit in the above equations we have

$$\mathcal{A}^1_0(t,p,q;\hbar) = \mathcal{L}_1(p,q)\mathrm{ns}(t,k), \tag{48a}$$

$$\mathcal{A}^2_0(t,p,q;\hbar) = \mathcal{L}_2(p,q)\mathrm{ds}(t,k), \tag{48b}$$

$$\mathcal{A}_0^3(t,p,q;\hbar) = \mathcal{L}_3(p,q)\mathrm{cs}(t,k), \tag{48c}$$

where $\mathcal{L}_1 = \frac{i}{2}p(q^2-1)$, $\mathcal{L}_2 = -\frac{1}{2}p(q^2+1)$ and $\mathcal{L}_3 = -ipq$. It is immediate to see that the functions $\mathcal{L}_j$ satisfy the algebra

$$\{\mathcal{L}_i, \mathcal{L}_j\}_P = \epsilon_{ijk}\mathcal{L}_k.$$

It is very interesting to see that Eqs. (48) are precisely the solutions obtained recently by Hashimoto et al. [24] as a hyper-Kähler metric on $\Sigma \times \mathbf{R}^2$ which is a particular solution of the Ashtekar-Jacobson-Smolin equations! [7]. Following the philosophy of Ref. [11], the above solution might be called *gravitational monopole.* Thus we have found a case where a solution of BPS monopole Nahm equations corresponds to a self-dual metric. This correspondence deserves a more careful study and we leave it for a separated paper.

The $SL(2;\mathbf{R})$ Magnetic Monopole

A similar situation as the above results occurs for the SL$(2;\mathbf{R})$ case. Now the solution of sl$(2,\mathbf{R})$ Nahm's equations (5) takes the form

$$A^j = A^j(t) = A^j_a(t)\tau_a, \qquad a = 1,2,3,$$

where

$$\tau_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{49}$$

satisfy

$$[\tau_1,\tau_2] = \tau_3, \qquad [\tau_2,\tau_3] = \tau_1, \qquad [\tau_3,\tau_1] = -\tau_2.$$

Following Ward's paper [21], the solutions of sl$(2;\mathbf{R})$ Nahm equation are

$$A^1 = k\mathrm{sn}(t,k), \qquad A^2 = k\mathrm{cn}(t,k), \qquad A^3 = \mathrm{dn}(t,k), \tag{50}$$

where ‘sn’, ‘cn’ and ‘dn’ are elliptic functions and k is a constant.

¿From Refs. [20,10] we define the Lie algebra homomorphism $\Phi : \mathrm{sl}(2;\mathbf{R}) \to \hat{\mathcal{U}}$ by

$$\Phi(\tau_1) = \hat{\mathcal{X}}_1 := \frac{i}{4}\Big(\frac{\hat{p}^2}{\hbar^2} + \frac{\delta}{\hat{q}^2} - \hat{q}^2\Big), \tag{51a}$$

$$\Phi(\tau_2) = \hat{\mathcal{X}}_2 := \frac{i}{4}\Big(\frac{\hat{p}^2}{\hbar^2} + \frac{\delta}{\hat{q}^2} + \hat{q}^2\Big), \tag{51b}$$

$$\Phi(\tau_3) = \hat{\mathcal{X}}_3 := \frac{i}{2}\Big(\frac{\hat{q}\hat{p}}{\hbar} - \frac{i}{2}\Big), \tag{51c}$$

where $\delta \in \Re$ is a constant.

Defining the functions $\mathcal{X}_a = \mathcal{X}_a(p,q)$ according to (42) and proceeding as above we get the real solution of the Moyal deformation of Nahm equation to be

$$\mathcal{A}^1(t,p,q;\hbar) = k\mathrm{sn}(t,k)[-\frac{\hbar^{-1}}{4}p^2 - \frac{\hbar\delta}{4}q^{-2} + \frac{\hbar}{4}q^2], \tag{52a}$$

$$\mathcal{A}^2(t,p,q;\hbar) = k\mathrm{cn}(t,k)[-\frac{\hbar^{-1}}{4}p^2 - \frac{\hbar\delta}{4}q^{-2} - \frac{\hbar}{4}q^2], \tag{52b}$$

$$\mathcal{A}^3(t,p,q;\hbar) = \mathrm{dn}(t,k)[-\frac{1}{2}pq]. \tag{52c}$$

One can see that once again this $\mathrm{sl}(2;\mathbf{R})$ BPS monopole defines the Moyal deformation but in this case the solution is not analytic in $\hbar$. Thus, in this case doesn’t exist an associated self-dual metric.

5. **Final Remarks**

In this paper we have found another example, different from self-dual gravity, where the WWM-formalism gets some solutions for large-N physical systems. In particular we have defined the Moyal deformation of Nahm equations and we have found the corresponding SU(∞) Nahm equations as the $\hbar \to 0$ limit instead of the usual one $N \to \infty$. After that we found some solutions of SU(∞) Nahm equations via the WWM-formalism. We have found strong evidence of the existence of the correspondence between the SU(2) BPS magnetic monopoles which are solutions of Nahm equations in agreement to [2] and hyper-Kähler metrics on $\Sigma \times \mathbf{R}^2$. We leave this important issue for a forthcoming paper. We think that Strachan geometry of multidimensional geometry [25] might be important to consider the whole and global issue where topological field theories and cobordism structure [26] are very important.

Remark

Some recent works on deformations of Nahm equations were given by Hoppe [27] and by Castro [28,29]. Hoppe shown that the existence of infinitely constants of motion is preserved under non-local deformations of some surfaces generated by time harmonic motions. In Ref. [28] Castro uses the WWM-formalism to construct Moyal deformations of the self-dual membrane in terms of the 3D Moyal deformation of the continuous Toda theory. Finally in Ref. [29] it was obtained the q-Moyal deformation of the self-dual membrane and very interesting explicit solutions were found. Embedding of Moyal-Toda chain into the SU(∞) Moyal-Nahm equation was given as well. We are grateful to the referee for pointing out the papers [27-29].

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